Title	On the Hamiltonian structures of the second and the fourth Painleve hierarchies, and the degenerate Garnier systems(Algebraic, Analytic and Geometric Aspects of Complex Differential Equations and their Deformations. Painleve Hierarchies)
Author(s)	KOIKE, Tatsuya
Citation	数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2007), B2: 99-127
Issue Date	2007-03
URL	http://hdl.handle.net/2433/174121
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

On the Hamiltonian structures of the second and the fourth Painlevé hierarchies, and the degenerate Garnier systems*

Tatsuya KOIKE†

0 Introduction.

The purpose of this paper is to show that each member of the (P_{J})-hierarchy $(\mathrm{J} = \mathrm{II}, \mathrm{IV})$ introduced by Gordoa-Joshi-Pickering ([GJP]) is equivalent to the restriction to an appropriate complex line of some degenerate Garnier system studied by Liu and Okamoto (cf. [L5], cf. also [LO1], [LO2], [L1], [L2], [L3], [L4]) and Kawamuko (cf. [Kwm4]. Its announcement appeared in [Kwm2], cf. also [Kwm1]). An interesting feature of our results is that the higher order Painlevé equations introduced by [GJP] and the degenerate Garnier system studied by Liu-Okamoto and Kawamuko have completely different origins; the former ones are found through non-isospectral scatterings, while the latter ones are derived from isomonodromic deformations of second order linear ordinary differential equations. Since the latter ones are expressed in the form of a Hamiltonian system with several time variables, we can thus find a Hamiltonian structure for the (P_{J})-hierarchy $(\mathrm{J} = \mathrm{II}, \mathrm{IV})$. This fact is crucially important in carrying out our Toulouse Project ([KT]), i.e., our program for the thorough understanding of the Painlevé hierarchy (P_{J}) $(\mathrm{J} = \mathrm{I}, \mathrm{II}, \mathrm{IV})$, because Takei ([T]) has recently established a neat way of constructing $2m$-parameter solutions of a non-linear ordinary differential equation of order $2m$ when it is given in the form of a Hamiltonian system.

The plan of this paper is as follows: In §1 we first recall the definition of the Painlevé hierarchies and degenerate Garnier systems, then we state main theorems of this article. In §2, we give a proof of our main theorem for the (P_{IV})-hierarchy. The proof of our main theorem for the (P_{II})-hierarchy is given in §3; we content ourselves with describing only its important points in order not to bore the reader, as it is basically the same as that for the case of the (P_{IV})-hierarchy. In Appendix A we discuss the relation between the compatibility conditions of a system of linear differential equations with 2 unknown functions and the compatibility conditions of a system of scalar differential equations derived from the above system of equations given in a matrix form. Although

*This work has been partially supported by JSPS Grants-in-Aid No. 18740071.

†Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto, 606-8502, Japan.

the argument is a straightforward one, the results are useful and important in our discussion. In Appendix B, a list of equations with a large parameter is given for future reference.

At the end of the introduction, I wish to express my sincere gratitude to Professors Takahiro Kawai, Takashi Aoki and Yoshitsugu Takei for helpful suggestions and stimulating discussions.

1 Main theorems.

1.1 The second and the fourth Painlevé hierarchies.

We discuss the second and the fourth Painlevé hierarchies given respectively in Definition 1.1 and Definition 1.2. As their names indicate, the first member $(P_{\mathrm{II}})_1$ (resp., $(P_{\mathrm{IV}})_1$) coincides with the traditional (i.e., the second order) second (resp., fourth) Painlevé equation. As we will show in [Ko], the hierarchies given below are equivalent to those introduced by [GJP]; the presentation of the hierarchies in the form of the first order systems is most suited for our purpose, i.e., for relating them with the Garnier systems.

Definition 1.1. For $m = 1, 2, \cdots$, $(P_{\mathrm{II}})_m$ is, by definition, the following system of non-linear differential equations:

$$(P_{\mathrm{II}})_m \quad \begin{cases} \dfrac{du_j}{dt} &= -2\Big[u_1 u_j + v_j + u_{j+1}\Big] + 2c_j u_1 \\ \dfrac{dv_j}{dt} &= 2\Big[v_1 u_j + v_{j+1} + w_j\Big] - 2c_j v_1 \end{cases} \qquad (1 \le j \le m) \tag{1.1}$$

$$\text{with} \quad u_{m+1} = \gamma t, \quad v_{m+1} = \kappa. \tag{1.2}$$

Here $\{u_j, v_j\}_{j=1}^m$ are the unknown functions, $\gamma\,(\neq 0)$, κ and $\{c_j\}_{j=1}^m$ are constants, and $\{w_n\}$ are polynomials of $\{u_j, v_j\}$ recursively defined by

$$w_n = \sum_{j=1}^{n-1} u_{n-j} w_j + \sum_{j=1}^{n} u_{n-j+1} v_j + \frac{1}{2} \sum_{j=1}^{n-1} v_{n-j} v_j - \sum_{j=1}^{n-1} c_{n-j} w_j. \tag{1.3}$$

Remark 1.1. (P_{II})-hierarchy defined here was called $(P_{\text{II-2}})$-hierarchy in [KKNT] to distinguish this hierarchy from another Painlevé hierarchy obtained through a similarity reduction of the modified KdV hierarchy, which was called $(P_{\text{II-1}})$-hierarchy in [KKNT]. Because we will not discuss the $(P_{\text{II-1}})$-hierarchy in this article, we call $(P_{\text{II-2}})$-hierarchy just (P_{II})-hierarchy. See [MM] and [Tks] for the Hamiltonian structure of $(P_{\text{II-1}})$-hierarchy and (P_{I})-hierarchy, respectively.

Remark 1.2. We slightly change the notations of [KKNT] so that we may make the correspondence of parameters clearer in relating degenerate Garnier systems and higher order Painlevé equations: the symbols g and δ will be now superseded respectively by 2γ and 2κ.

Remark 1.3. Without loss of generality, we can choose c_1 to be zero and fix γ as an arbitrary nonzero constant by using scalings and translations of the independent and unknown variables. In our main theorem (Theorem 1.3) we choose $c_1 = 0$ and $\gamma = 2^{-m-2}$.

Remark 1.4. (i) First two members of $\{w_n\}$ are given as follows:

$$
\begin{aligned}
w_1 &= u_1 v_1, && (1.4)\\
w_2 &= (u_1 - c_1)w_1 + \frac{1}{2}{v_1}^2 + u_1 v_2 + u_2 v_1 \\
&= {u_1}^2 v_1 - c_1 u_1 v_1 + u_1 v_2 + u_2 v_1 + \frac{1}{2}{v_1}^2. && (1.5)
\end{aligned}
$$

(ii) $(P_{\mathrm{II}})_1$; by setting $c_1 = 0$, we find $u = -2u_1$ satisfies

$$\frac{d^2u}{dt^2} = 2u^3 + 8\gamma t u + 8\kappa + 4\gamma. \tag{1.6}$$

$(P_{\mathrm{II}})_2$; by setting $c_2 = 0$, we find $u = -2u_1$ satisfies

$$
\begin{aligned}
u'''' = \frac{1}{2u^2}\Big[& 4uu'u'' + 3u(u'')^2 - 4(u')^2 u'' \\
& + (10u^4 + 32\gamma t u)u'' + (5u^3 - 32\gamma t)(u')^2 \\
& + 32\gamma u u' \\
& - 5u^7 - 24c_2 u^5 - 32\gamma t u^4 - (16{c_2}^2 + 48\gamma + 96\kappa)u^3 + 64\gamma^2 t^2 u\Big],
\end{aligned} \tag{1.7}
$$

where $'$ stands for d/dt.

Definition 1.2. For $m = 1, 2, \cdots$, $(P_{\mathrm{IV}})_m$ is, by definition, the following system of non-linear differential equations:

$$
(P_{\mathrm{IV}})_m \quad \left\{
\begin{aligned}
\frac{du_j}{dt} &= -2\Big[u_1 u_j + v_j + u_{j+1}\Big] + 2c_j u_1 \\
\frac{dv_j}{dt} &= 2\Big[v_1 u_j + v_{j+1} + w_j\Big] - 2c_j v_1
\end{aligned}
\right. \qquad (1 \le j \le m) \tag{1.8}
$$

$$\text{with} \quad u_{m+1} = -(\gamma t u_1 + \theta_1 + \frac{1}{2}\gamma), \tag{1.9}$$

$$v_{m+1} = -w_m - \gamma t v_1 - \frac{(v_m - \theta_1)^2 - {\theta_2}^2}{2(u_m - \gamma t - c_m)}. \tag{1.10}$$

Here $\{u_j, v_j\}_{j=1}^m$ are the unknown functions, $\gamma\,(\neq 0)$, θ_1, θ_2 and $\{c_j\}_{j=1}^m$ are constants, and $\{w_n\}$ are polynomials of $\{u_j, v_j\}$ recursively defined by (1.3).

Remark 1.5. In parallel with the (P_{II})-hierarchy, we can choose $c_m = 0$ and fix γ to an arbitrary nonzero constant without loss of generality. Later in Theorem 1.4, we will choose $\gamma = 2^{-m}$. Constants γ, θ_1 and θ_2 in $(P_{\mathrm{IV}})_m$ defined here are related to constants $\{g_m, \kappa, \theta\}$ used in [GJP] by $g_m = 2\gamma$, $\kappa = 2\theta_1 + \gamma$, $\theta = 4\theta_2$.

Remark 1.6. $(P_{\mathrm{IV}})_1$; $y = -2u_1 + 2c_1 + 2\gamma t$ satisfies

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^2 - 4\gamma t y^2 + (2\gamma^2 t^2 - 2\gamma - 4\theta_1)y - \frac{8{\theta_1}^2}{y}.$$

1.2 The degenerate Garnier system studied by Liu and Okamoto.

Liu and Okamoto studied the holonomic (i.e., isomonodromic) deformation of a linear ordinary differential equation of the form

$$\frac{d^2y}{dz^2} + p_1(z,t)\frac{dy}{dz} + p_2(z,t)y = 0, \tag{1.11}$$

where

$$p_1(z,t) = -A_g(z,t) - \sum_{k=1}^{g} \frac{1}{z-\lambda_k}, \quad A_g(z,t) = 2z^{g+1} + \sum_{j=1}^{g} j t_j z^{j-1}, \tag{1.12}$$

$$p_2(z,t) = -(2\alpha+1)z^g - 2\sum_{j=1}^{g} h_j z^{g-j} + \sum_{k=1}^{g} \frac{\mu_k}{z-\lambda_k} \tag{1.13}$$

for $g = 1, 2, \cdots$. (See [L5] for the result for an arbitrary $g \geq 1$. See also [L1]. The cases for small g were studied in [LO1], [L2] and [L3].) Here α is a constant, $t = (t_1, \cdots, t_g)$ a deformation parameter, and $\{\lambda_j, \mu_j\}$ are functions of t. The equation (1.11) contains g regular singular points located at $z = \lambda_j$ $(1 \leq j \leq g)$, which are with characteristic exponents 0 and 2, and one irregular singular point at $z = \infty$.

Liu and Okamoto assumed that α is not a half integer, and that all of $z = \lambda_j$ for $1 \leq j \leq g$ are non-logarithmic singularities. We note that the latter assumption enables us to determine $\{h_j\}$ uniquely as rational functions of $\{\lambda_j, \mu_j, t_j\}$. In fact the Frobenius method enables one to fix $\{h_j\}$ as follows ([L5, Proposition 2.1, p.570]):

$$h_j = \frac{1}{2}\sum_{k=1}^{g} \left[N_k N^{j,k} {\mu_k}^2 - U_{j,k}\mu_k - (2\alpha+1)N_k N^{j,k} {\lambda_k}^g\right] \tag{1.14}$$

for $j = 1, 2, \cdots, g$, where

$$N_k = \frac{1}{\Lambda'(\lambda_k)} \quad \text{with} \quad \Lambda(z) = \prod_{j=1}^{g}(z-\lambda_j) \quad \text{and} \quad \Lambda'(z) = \frac{d\Lambda}{dz}, \tag{1.15}$$

$$N^{j,k} = (-1)^{j-1} e^{(k)}_{j-1}(\lambda), \tag{1.16}$$

$$U_{j,k} = N_k N^{j,k} A_g(\lambda_k, t) + \sum_{\substack{l=1,2,\cdots,g \\ l \neq k}} \frac{N_k N^{j,k} + N_l N^{j,l}}{\lambda_k - \lambda_l}, \tag{1.17}$$

with $e^{(k)}_l$ being the l-th symmetric polynomial of $\{\lambda_j;\, j \neq k\}$. (We set $e^{(k)}_0 = 1$ as a convention.)

Liu and Okamoto then proved the following:

Theorem 1.1. *([L5, Main theorem (p.560), Propositoin 1.1 (p.568)])*
We assume that $\{\lambda_j\}$ *are non-logarithmic singular points of* (1.11)*, that* $\{\lambda_j(t), \mu_j(t)\}$ *are functions of* $t = (t_1, \ldots, t_g)$*, that* α *is a constant and not a half-integer, and that all singular points of* (1.11) *are distinct. Then the following conditions (i), (ii) and (iii) are equivalent:*

(i) Eq. (1.11) *admits a monodoromy preserving deformation with respect to* t *in the sense of [JMU].*

(ii) There exist $\{\mathcal{A}_k, \mathcal{B}_k\}_{k=1}^{g}$*, which are rational functions in* z*, such that* (1.11) *and the following equations for* $k = 1, 2, \cdots, g$ *are completely integrable:*

$$\frac{\partial y}{\partial t_k} = \mathcal{A}_k \frac{\partial y}{\partial z} + \mathcal{B}_k y. \tag{1.18}$$

(iii) $\{\lambda_j, \mu_j\}$ *satisfy the following completely integrable Hamiltonian systems:*

$$\frac{\partial \lambda_j}{\partial t_k} = \frac{\partial H_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\frac{\partial H_k}{\partial \lambda_j} \tag{1.19}$$

for $j, k = 1, 2, \cdots, g$*. Here*

$$H_j = 2\sum_{i=0}^{j-1} a_{i+1}(t)(h_{j-i} + T^*_{j-i}) \quad (1 \leq j \leq g), \tag{1.20}$$

where $\{a_j(t)\}$ *and* $\{T^*_l\}$ *are given by*

$$a_1(t) = \frac{1}{2},\ a_2(t) = 0,\ a_{r+1}(t) = -\frac{1}{2}\sum_{j=1}^{r-1} T_{g+j-r+1} a_j(t)\ (2 \leq r \leq g-1), \tag{1.21}$$

$$T_l = \begin{cases} lt_l & (1 \leq l \leq g) \\ 0 & (otherwise) \end{cases}, \tag{1.22}$$

$$T^*_l = \frac{1}{4}(l-1)T_{g-l+2} + \frac{1}{8}\sum_{j=1}^{g} T_j T_{g-j-l+2} \quad (1 \leq l \leq g). \tag{1.23}$$

These degenerate Garnier systems (1.19) are called A_g*-systems*.

Remark 1.7. Functions $\{\mathcal{A}_j\}$ are explicitly determined in [L5, Proposition 2.1, p.572]. In particular [L5] gives

$$\mathcal{A}_1 = \frac{1}{2\Lambda(z)} = \frac{1}{2(z-\lambda_1)\cdots(z-\lambda_g)}. \tag{1.24}$$

Once $\{\mathcal{A}_j\}$ are given, we can determine $\{\mathcal{B}_j\}$ by using the compatibility condition of (1.11) and (1.18). (See (A.12) and (A.13) in Appendix A.) In fact we can determine $\mathcal{B}_j$ by (A.15) with $x = z$, $t = t_j$, $p = p_1$, $q = p_2$, $\mathcal{A} = \mathcal{A}_j$ and $\mathcal{B} = \mathcal{B}_j$. We also note that for each j, (1.19) is known to be equivalent to (A.16) ([L5, §5]).

1.3 The degenerate Garnier system studied by Kawamuko.

Kawamuko considered the holonomic deformation of the following linear ordinary differential equation of the second order in [Kwm4]. (Its announcement was given in [Kwm2]. Cf. [Kwm1].)

$$\frac{d^2y}{dz^2} + p_1(z,t)\frac{dy}{dz} + p_2(z,t)y = 0, \tag{1.25}$$

where

$$p_1(z,t) = -\sum_{k=0}^{g+1} t_k z^{k-1} - \sum_{k=1}^{g} \frac{1}{z-\lambda_k} \quad (t_{g+1} = 1,\, t_0 = \kappa_0 - 1), \tag{1.26}$$

$$p_2(z,t) = \kappa_\infty z^{g-1} - \frac{1}{z}\sum_{k=1}^{g} h_{g+1-k} z^{k-1} + \sum_{k=1}^{g} \frac{\lambda_k \mu_k}{z(z-\lambda_k)} \tag{1.27}$$

for $g = 1, 2, \cdots$. Here κ_0 and κ_∞ are constants, $t = (t_1, \cdots, t_g)$ is a deformation parameter, and $\{(\lambda_j, \mu_j)\}$ are functions of t. The equation (1.25) contains g regular singular points located at $z = \lambda_j$ $(1 \le j \le g)$ with characteristic exponents being 0 and 2, one regular singular point at $z = 0$ with characteristic exponents 0 and κ_0, and one irregular singular point at $z = \infty$.

He also assumed that neither κ_0 nor $2\kappa_\infty - \kappa_0$ is an integer, and that any of $z = \lambda_j$ is non-logarithmic singular point. As in the case of A_g-systems discussed in §1.2, $\{h_j\}$ can be uniquely determined as rational functions of $\{\lambda_j, \mu_j, t_j\}$ as follows:

$$\begin{aligned} h_{j+1} = (-1)^j \sum_{l=1}^{g} \frac{e_j^{(l)}}{\Lambda'(\lambda_l)} \left\{ \lambda_l \mu_l^2 - \left(\sum_{k=1}^{g+1} t_k \lambda_l^k + \kappa_0 \right) \mu_l + \kappa_\infty \lambda_l^g \right\} \\ - \sum_{l=1}^{g} \frac{\mu_l}{\Lambda'(\lambda_l)} \sum_{k=0}^{j-1} (-1)^k e_k^{(l)} \lambda_l^{j-k} \end{aligned} \tag{1.28}$$

for $j = 0, 1, \cdots, g-1$. Here $\Lambda(x) = (x-\lambda_1)\cdots(x-\lambda_g)$ and $e_l^{(k)}$ is the l-th symmetric polynomial of $\{\lambda_j;\, j \ne k\}$. (Note that $e_0^{(k)} = 1$ by convention.)

Kawamuko then obtained the following:

Theorem 1.2. *([Kwm4, Theorem 1.1 (p.3)])*
We assume that $\{\lambda_j\}$ are non-logarithmic singular points of (1.25), that $\{\lambda_j, \mu_j\}$ are functions of $t = (t_1, \cdots, t_g)$, that κ_0 and κ_∞ are constants with respect to t such that κ_0 and $2\kappa_\infty - \kappa_0$ are not integers, and that all singular points in (1.25) are distinct. Then the following three conditions (i), (ii) and (iii) are equivalent:

(i) Eq. (1.25) admits a monodoromy preserving deformation with respect to t in the sense of [JMU].

(ii) There exist $\{\mathcal{A}_k, \mathcal{B}_k\}_{k=1}^g$, which are rational functions in z, such that (1.25) and the following equations for $k = 1, 2, \cdots, g$ are completely integrable:

$$\frac{\partial y}{\partial t_j} = \mathcal{A}_k \frac{\partial y}{\partial z} + \mathcal{B}_k y \quad (1 \le k \le g). \tag{1.29}$$

(iii) $\{\lambda_j, \mu_j\}$ *satisfy the following completely integrable Hamiltonian systems:*

$$\frac{\partial \lambda_j}{\partial t_k} = \frac{\partial H_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\frac{\partial H_k}{\partial \lambda_j} \qquad (1 \le j, k \le g). \tag{1.30}$$

Here

$$H_j = \frac{1}{j} \sum_{k=1}^{j} T_{j-k} h_k \quad (1 \le j \le g) \tag{1.31}$$

and $\{T_j\}$ *are defined through the following relation:*

$$(1 + t_g \xi + t_{g-1} \xi^2 + \cdots + t_1 \xi^g)^{-1} = \sum_{j=0}^{\infty} T_j \xi^j. \tag{1.32}$$

Kawamuko considered the degenerate Garnier system defined by (1.30) as the fourth Painlevé equations with several variables. In the following we call these systems as *Kawamuko's systems.*

Remark 1.8. Functions $\{\mathcal{A}_j\}$ are explicitly determined in [Kwm4, (3.5) and (3.8)]. In particular we find

$$\mathcal{A}_1 = \frac{z}{\Lambda(z)} = \frac{z}{(z-\lambda_1)\cdots(z-\lambda_g)}. \tag{1.33}$$

Once $\{\mathcal{A}_j\}$ are obtained, as is explained in Remark 1.7, we can determine $\{\mathcal{B}_j\}$ by (A.15). Furthermore (1.30) is known to be equivalent to (A.16) for each j with $x = z$, $t = t_j$, $p = p_1$, $q = p_2$, $\mathcal{A} = \mathcal{A}_j$ ([Kwm4, §4]).

1.4 Main Theorems.

We now state our main theorems. They claim that the (P_{II})-hierarchy and the (P_{IV})-hierarchy defined in §1.1 are respectively equivalent to the restriction to an appropriate complex line of a degenerate Garnier system studied by Liu-Okamoto (§1.2) and that studied by Kawamuko (§1.3). To state our main theorems we introduce the following polynomials $U(x)$, $V(x)$ and $C(x)$ for $\{u_j\}_{j=1}^m$, $\{v_j\}_{j=1}^m$ and $\{c_j\}_{j=1}^m$ in §1.1:

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \quad V(x) = \sum_{j=1}^{m} v_j x^{m-j}, \quad C(x) = \sum_{j=1}^{m} c_j x^{m-j}. \tag{1.34}$$

Theorem 1.3. *Let* $\{u_j, v_j\}_{j=1}^m$ *be a solution of* $(P_{II})_m$ *given by* (1.1) *with* $c_1 = 0$, *and let* $K = K(\lambda_j, \mu_j, t)$ *be a rational function of* $\{\lambda_j, \mu_j, t\}$ *defined as follows:*

$$K(\lambda_j, \mu_j, t) = H_1(\lambda_j, \mu_j, t_1, \cdots, t_m)\Bigg|_{\begin{cases} t_1 = t, \\ t_k = 2^{m-k+3} c_{m-k+2}/k \quad (2 \le k \le m) \end{cases}}, \tag{1.35}$$

where H_1 is the Hamiltonian of A_g-system defined by (1.20) *with $g = m$. If the equation $U(x)+C(x) = 0$ has no double roots with respect to x, then $\{\lambda_j, \mu_j\}$ determined by the relations*

$$U(x) + C(x) = \prod_{j=1}^{m}(x - \frac{1}{2}\lambda_j), \quad \mu_j = -2^{m+1}V(\frac{1}{2}\lambda_j) \tag{1.36}$$

solve the following Hamiltonian system

$$\frac{d\lambda_j}{dt} = \frac{\partial K}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\frac{\partial K}{\partial \lambda_j} \quad (1 \leq j \leq m). \tag{1.37}$$

Here we have assumed the following relations among the constants:

$$\gamma = 2^{-m-2}, \quad \kappa = -2^{-m-2}(\alpha + \frac{1}{2}). \tag{1.38}$$

Conversely if $\{\lambda_j, \mu_j\}$ satisfy (1.37) *with K the function defined by* (1.35) *and if $\lambda_j \neq \lambda_k$ ($j \neq k$) hold, then $\{u_j, v_j\}$ determined by* (1.36) *is a solution of $(P_{II})_m$ and $U(x) + C(x) = 0$ has no double roots with respect to x. This correspondence of solutions between $(P_{II})_m$ and* (1.37) *is one-to-one up to a permutation of the pairs $\{(\lambda_j, \mu_j)\}$.*

Remark 1.9. The first equation of (1.36) reduces to

$$u_j - c_j = \frac{(-1)^{j+1}}{2^j} e_j(\lambda), \tag{1.39}$$

where $e_j(\lambda)$ is the j-th symmetric polynomial of $\{\lambda_l\}$.

Remark 1.10. The symmetric polynomials $\{e_j(\lambda)\}$ are a part of canonical variables employed by Liu ([L1], see also [LO2] and [L4] for $g = 3, 4$) to find a polynomial Hamiltonian. In this sense we may say $\{u_j\}$ are "good" variables.

Remark 1.11. It may be worth emphasizing that the arbitrary constants $\{c_j\}_{j=1}^{m-1}$ in the formulation of [GJP] are related to the variables $\{t_l\}_{l=2}^{m}$ in the Garnier system.

Theorem 1.4. *Let $\{u_j, v_j\}_{j=1}^m$ be a solution of $(P_{IV})_m$ given by* (1.8), *and let $K = K(\lambda_j, \mu_j, t)$ be a rational function of $\{\lambda_j, \mu_j, t\}$ defined by the following:*

$$K(\lambda_j, \mu_j, t) = H_1(\lambda_j, \mu_j, t_1, \cdots, t_m)\Bigg|_{\begin{cases} t_1 = t + 2^m c_m, \\ t_k = 2^{m-k+1}c_{m-k+1} \quad (2 \leq k \leq m) \end{cases}}, \tag{1.40}$$

where H_1 is the Hamiltonian of Kawamuko's systems defined by (1.31) *with $g = m$. If $U(x) + C(x) + \gamma t = 0$ has no double roots with respect to x, where $U(x)$ and $C(x)$ are defined by* (1.34), *then $\{\lambda_j, \mu_j\}$ determined by the relations*

$$U(x) + C(x) + \gamma t = \prod_{j=1}^{m}(x - \frac{1}{2}\lambda_j), \quad \lambda_j\mu_j = -2^m\big[V(\frac{1}{2}\lambda_j) - \theta_1 - \theta_2\big] \tag{1.41}$$

solve the following Hamiltonian system

$$\frac{d\lambda_j}{dt} = \frac{\partial K}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\frac{\partial K}{\partial \lambda_j} \quad (1 \le j \le m). \tag{1.42}$$

Here we have assumed the following relations among constants:

$$\gamma = 2^{-m}, \quad \theta_1 = -2^{-m-1}(\kappa_0 - 2\kappa_\infty), \quad \theta_2 = 2^{-m-1}\kappa_0. \tag{1.43}$$

Conversely if $\{\lambda_j, \mu_j\}$ *satisfy* (1.42) *with* K *the function defined by* (1.40)*, and if* $\lambda_j \neq \lambda_k$ $(j \neq k)$ *hold, then* $\{u_j, v_j\}$ *determined by* (1.41) *is a solution of* $(P_{IV})_m$ *and* $U(x) + C(x) + \gamma t = 0$ *has no double roots with respect to* x*. This correspondence of solutions between* $(P_{IV})_m$ *and* (1.42) *is one-to-one up to a permutation of the pairs* $\{(\lambda_j, \mu_j)\}$*.*

Remark 1.12. Note that the first equation of (1.41) is equivalent to

$$u_j - c_j = \frac{(-1)^{j+1}}{2^j} e_j(\lambda) \quad (1 \le j \le m-1), \tag{1.44}$$

$$u_m - c_m - \gamma t = \frac{(-1)^{m+1}}{2^m} e_m(\lambda), \tag{1.45}$$

where $e_j(\lambda)$ is the j-th symmetric polynomial of $\{\lambda_l\}$.

Remark 1.13. In [Kwm3] (cf. [Kwm1]) Kawamuko's system was canonically transformed to a system whose Hamiltonians are polynomials of the canonical variables. There the symmetric polynomials $\{e_j(\lambda)\}$ are again chosen as a part of the canonical variables. In this sense $\{u_j\}$ are "good" variables.

In the following sections, we give the proofs of these two theorems. Since we can prove them in the same manner, we will mainly discuss a proof of Theorem 1.4 in §2, and give only an outline of the proof of Theorem 1.3 in §3.

2 Proof of the main theorem for P_{IV}-hierarchy.

In this section we first give the Lax pair (2.1) of (P_{IV})-hierarchy (§2.1). In §2.2, we transform this Lax pair to a system of scalar equations (2.23a) and (2.23b) after some transformation of the unknown function. Incidentally we note that the compatibility condition of this system of scalar equations also gives $(P_{\mathrm{IV}})_m$. Then, in §2.3, we compare these scalar equations with the linear equations (1.25) associated with Kawamuko's system to find that the compatibility condition of these scalar equations (2.23a) and (2.23b) gives (1.42), completing the proof of Theorem 1.4.

2.1 Lax pair for $(P_{\mathrm{IV}})_m$.

The Lax pair (2.1) below plays a key role in the proof of Theorem 1.4:

Theorem 2.1. *$(P_{IV})_m$ is equivalent to the compatibility condition of the following equations:*

$$\gamma x \frac{\partial \vec{\psi}}{\partial x} = A\vec{\psi}, \quad \frac{\partial \vec{\psi}}{\partial t} = B\vec{\psi}, \tag{2.1}$$

where

$$A = \begin{pmatrix} -\Big[x^{m+1} + V + xC(x) + \gamma xt - \theta_1\Big] & U + C(x) + \gamma t \\ -2\Big[xV + W + v_{m+1} + \gamma t v_1\Big] & x^{m+1} + V + xC(x) + \gamma xt - \theta_1 \end{pmatrix}, \tag{2.2}$$

$$B = \begin{pmatrix} -(x+u_1) & 1 \\ -2v_1 & x+u_1 \end{pmatrix}, \tag{2.3}$$

where

$$W(x) = \sum_{j=1}^{m} w_j x^{m-j}, \tag{2.4}$$

for $\{w_j\}$ given by (1.3) and $U(x)$, $V(x)$ and $C(x)$ are those given by (1.34).

The above Lax pair (2.1) is essentially the same as the Lax pair for the corresponding member of the fourth Painlevé hierarchy introduced by P. R. Gordoa, N. Joshi and A. Pickering ([GJP]), though they look quite different. Actually the above Lax pair can be readily obtained from the Lax pair that [GJP] uses, through the replacement of the unknown functions (u, v) of the fourth Painlevé hierarchy of [GJP] by our unknown functions (u_j, v_j) of $(P_{\mathrm{IV}})_m$. We note that the same replacement of unknown functions enables us to show that our (P_{IV})-hierarchy is equivalent to the fourth Painlevé hierarchy of [GJP]. (See [Ko] for the details.)

Let Δ_j $(j = 1, 2, 3)$ be given by

$$\frac{\partial A}{\partial t} - \gamma x \frac{\partial B}{\partial x} + AB - BA = \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & -\Delta_1 \end{pmatrix} \tag{2.5}$$

for matrices A and B defined by (2.2) and (2.3). Then $\Delta_1 = \Delta_2 = \Delta_3 = 0$ gives the compatibility condition of (2.1). Therefore Theorem 2.1 is a consequence of the following proposition:

Proposition 2.1. *Let Δ_j $(j = 1, 2, 3)$ be given by (2.5). Then*

(i) $\Delta_1 = \Delta_2 = 0$ holds if and only if $(P_{IV})_m$ defined by (1.8) holds.

(ii) If $\{u_j, v_j\}$ satisfy $(P_{IV})_m$, then

$$(a) \quad \frac{dw_j}{dt} - v_1 \frac{du_j}{dt} - u_1 \frac{dv_j}{dt} = 0 \quad \textit{for} \quad 1 \le j \le m.$$

$$(b) \quad \frac{d}{dt}\Big[v_{m+1} + \gamma t v_1\Big] + \gamma v_1 = 0.$$

(iii) $\Delta_1 = \Delta_2 = 0$ *implies* $\Delta_3 = 0$.

Remark 2.1. The assertion (ii) plays an important role in confirming the assertion (iii).

Proof. (i) A straightforward computation shows

$$\Delta_1 = -\frac{dV}{dt} - 2v_1\Big[U + C(x) + \gamma t\Big] + 2\Big[xV + W + v_{m+1} + \gamma t v_1\Big], \tag{2.6}$$

$$\begin{aligned} \Delta_2 = \frac{dU}{dt} + \gamma - 2\Big[x^{m+1} + V + xC(x) + \gamma xt - \theta_1\Big] \\ + 2(x+u_1)\Big[U + C(x) + \gamma t\Big], \end{aligned} \tag{2.7}$$

$$\begin{aligned} \Delta_3 = -2\frac{d}{dt}\Big[xV + W + v_{m+1} + \gamma t v_1\Big] \\ + 4(x+u_1)\Big[xV + W + v_{m+1} + \gamma t v_1\Big] \\ - 4v_1\Big[x^{m+1} + V + xC(x) + \gamma xt - \kappa_1\Big]. \end{aligned} \tag{2.8}$$

Then it follows from (2.6) and (2.7) that

$$\Delta_1 = \sum_{j=1}^{m}\left[-\frac{dv_j}{dt} + 2v_1u_j + 2w_j + 2v_{j+1} - 2c_jv_1\right]x^{m-j}, \tag{2.9}$$

$$\Delta_2 = \sum_{j=1}^{m}\left[-\frac{du_j}{dt} - 2u_1u_j - 2v_j - 2u_{j+1} + 2c_ju_1\right]x^{m-j}. \tag{2.10}$$

Hence $\Delta_1 = \Delta_2 = 0$ is equivalent to (1.8).
(ii) We prove (a) by the induction on j. By differentiating (1.3) with respect to t, we obtain

$$\begin{aligned} \frac{dw_n}{dt} = \sum_{j=1}^{n-1}\frac{du_{n-j}}{dt}w_j + \sum_{j=1}^{n-1}u_{n-j}\frac{dw_j}{dt} + \sum_{j=1}^{n}\frac{du_{n-j+1}}{dt}v_j \\ + \sum_{j=1}^{n}u_{n-j+1}\frac{dv_j}{dt} + \sum_{j=1}^{n-1}v_{n-j}\frac{dv_j}{dt} - \sum_{j=1}^{n-1}c_{n-j}\frac{dw_j}{dt}. \end{aligned} \tag{2.11}$$

Then by a straightforward though somewhat lengthy computation, we find

$$\begin{aligned} &\frac{dw_n}{dt} - u_1\frac{dv_n}{dt} - v_1\frac{du_n}{dt} \\ &= \sum_{j=1}^{n-1}\frac{du_{n-j}}{dt}\left[-\frac{1}{2}\frac{dv_j}{dt} + v_1u_j + v_{j+1} + w_j - c_jv_1\right] \\ &\quad + \sum_{j=1}^{n-1}\left[\frac{1}{2}\frac{du_{n-j}}{dt} + u_1u_{n-j} + v_{n-j} + u_{n-j+1} - c_ju_1\right]\frac{dv_j}{dt} \end{aligned}$$

$$+\sum_{j=1}^{n-1}(u_{n-j}-c_{n-j})\Big[\frac{dw_j}{dt}-u_1\frac{dv_j}{dt}-v_1\frac{du_{n-j}}{dt}\Big]. \tag{2.12}$$

Note that we have not used (1.8) up to this point. Now we use (1.8) to obtain

$$\frac{dw_n}{dt}-u_1\frac{dv_n}{dt}-v_1\frac{du_n}{dt}=\sum_{j=1}^{n-1}(u_{n-j}-c_{n-j})\Big[\frac{dw_j}{dt}-u_1\frac{dv_j}{dt}-v_1\frac{du_{n-j}}{dt}\Big]. \tag{2.13}$$

Then the induction on j validates the assertion (a). Next we prove the relation (b). Let

$$\Phi=\frac{(v_m-\theta_1)^2-{\theta_2}^2}{2(u_m-\gamma t-c_m)}. \tag{2.14}$$

Then

$$\frac{d}{dt}\Big[v_{m+1}+\gamma t v_1\Big]+\gamma v_1=-\frac{dw_m}{dt}-\frac{d\Phi}{dt}+\gamma v_1 \tag{2.15}$$

follows from (1.10). We now compute dw_m/dt and $d\Phi/dt$. First (a) and (1.8) imply

$$\begin{aligned}\frac{dw_m}{dt}&=v_1\frac{du_m}{dt}+u_1\frac{dv_m}{dt}\\&=-2v_1\big\{u_1u_m+v_m+u_{m+1}\big\}+2u_1v_1c_m\\&\qquad+2u_1\big\{v_1u_m+v_{m+1}+w_m\big\}-2u_1v_1c_m\\&=-2v_1(v_m+u_{m+1})+2v_1(v_{m+1}+w_m)\\&=-2v_1v_m+2\theta_1v_1+\gamma v_1-2u_1\Phi.\end{aligned} \tag{2.16}$$

On the other hand (1.8) entails the following:

$$\begin{aligned}\Phi&=\frac{v_m-\kappa_1}{u_m-\gamma t-c_m}\frac{dv_m}{dt}-\frac{\Phi}{u_m-\gamma t-c_m}\left\{\frac{du_m}{dt}-\gamma\right\}\\&=2\frac{v_m-\theta_1}{u_m-\gamma t-c_m}\Big[v_1u_m+v_{m+1}+w_m-c_mv_1\Big]\\&\qquad+2\frac{\Phi}{u_m-\gamma t-c_m}\Big[u_1u_m+v_m+u_{m+1}-c_mu_1+\frac{1}{2}\gamma\Big]\\&=2\frac{v_m-\kappa_1}{u_m-\gamma t-c_m}\Big[v_1(u_m-\gamma t-c_m)-\Phi\Big]\\&\qquad+2\frac{\Phi}{u_m-\gamma t-c_m}\Big[u_1(u_m-\gamma t-c_m)+v_m-\kappa_1\Big]\\&=2v_1(v_m-\kappa_1)+2u_1\Phi.\end{aligned} \tag{2.17}$$

Hence (2.15), (2.16) and (2.17) prove (b).
(iii) A straightforward computation shows

$$\frac{1}{2}\Delta_3=(x+u_1)\Delta_1+v_1\Delta_2$$

$$
\begin{aligned}
&- v_1\frac{dU}{dt} + u_1\frac{dV}{dt} - \frac{dW}{dt} - \frac{d}{dt}\Big[v_{m+1} + \gamma t v_1\Big] - \gamma v_1 \\
&= (x+u_1)\Delta_1 + v_1\Delta_2 \\
&- \sum_{j=1}^{m}\left\{\frac{dw_j}{dt} - v_1\frac{du_j}{dt} - u_1\frac{dv_j}{dt}\right\}x^{m-j} - \frac{d}{dt}\Big[v_{m+1} + \gamma t v_1\Big] - \gamma v_1.
\end{aligned} \tag{2.18}
$$

Since $\Delta_1 = \Delta_2 = 0$ implies (1.8), it follows from (ii) that $\Delta_3 = 0$. $\square$

2.2 Another form of the Lax pair for $(P_{\mathbf{IV}})_m$.

In order to relate $(P_{\mathrm{IV}})_m$ with Kawamuko's system, we first replace the unknown function $\vec{\psi}$ of (2.1) by

$$
\vec{\psi} = \exp\left[-\int^x \frac{x^{m+1} + xC(x) + \gamma x t + \theta_2}{\gamma x}dx\right]\vec{\varphi}. \tag{2.19}
$$

Then the new unknown function $\vec{\varphi}$ satisfies

$$
\gamma x\frac{\partial\vec{\varphi}}{\partial x} = \widetilde{A}\vec{\varphi}, \quad \frac{\partial\vec{\varphi}}{\partial t} = \widetilde{B}\vec{\varphi} \tag{2.20}
$$

with

$$
\begin{aligned}
\widetilde{A} &= A + (x^{m+1} + C(x) + \gamma x t + \theta_2)I_2 \\
&= \begin{pmatrix} -(V - \theta_1 - \theta_2) & U + C(x) + \gamma t \\ -2(xV + W + v_{m+1} + \gamma t v_1) & \begin{array}{r} 2x^{m+1} + V + 2xC(x) \\ +2\gamma x t - \theta_1 + \theta_2 \end{array} \end{pmatrix},
\end{aligned} \tag{2.21}
$$

$$
\widetilde{B} = B + xI_2 = \begin{pmatrix} -u_1 & 1 \\ -2v_1 & 2x + u_1 \end{pmatrix}, \tag{2.22}
$$

where I_2 is the 2×2 identity matrix.
By using a standard procedure (cf. Proposition A.2 in Appendix A), we then find that the first component of $\vec{\varphi}$ satisfies

$$
\left(\frac{\partial^2}{\partial x^2} + q_1(x)\frac{\partial}{\partial x} + q_2(x)\right)\varphi = 0, \tag{2.23a}
$$

$$
\frac{\partial\varphi}{\partial t} = \mathcal{C}\frac{\partial\varphi}{\partial x} + \mathcal{D}\varphi, \tag{2.23b}
$$

where

$$
q_1 = -\frac{1}{\gamma x}\operatorname{tr}\widetilde{A} + \frac{1}{x} - \frac{1}{\widetilde{A}_{1,2}}\frac{\partial\widetilde{A}_{1,2}}{\partial x}, \tag{2.24}
$$

$$
q_2 = \frac{1}{\gamma^2 x^2}\det\widetilde{A} + \frac{1}{\gamma x}\frac{\partial V}{\partial x} - \frac{V - \theta_1 - \theta_2}{\gamma x\widetilde{A}_{1,2}}\frac{\partial\widetilde{A}_{1,2}}{\partial x}, \tag{2.25}
$$

$$\mathcal{C} = \frac{\gamma x}{\widetilde{A}_{1,2}}, \qquad \mathcal{D} = \frac{V - \theta_1 - \theta_2}{\widetilde{A}_{1,2}} - u_1. \tag{2.26}$$

Here $\widetilde{A}_{1,2}$ denotes the $(1,2)$-component of the matrix $\widetilde{A}$, i.e., $U + C(x) + \gamma$.
For the system of scalar equations (2.23a) and (2.23b), we can prove the following:

Proposition 2.2. *We assume that $\widetilde{A}_{1,2} = U + C(x) + \gamma t = 0$ has no double roots with respect to x. Then* (2.23a) *and* (2.23b) *are compatible if and only if* (2.20) *are compatible.*

It is clear that if (2.20) is compatible, then so are (2.23a) and (2.23b). Although its converse may also be obvious to an expert, we will give its proof at the end of Appendix A as a corollary of the general properties of the compatibility conditions.

To relate (2.23a) with (1.25), we first factorize $\widetilde{A}_{1,2}$ as

$$\widetilde{A}_{1,2} = U + C(x) + \gamma t = \prod_{j=1}^{m}(x - \frac{1}{2}\lambda_j). \tag{2.27}$$

(As we see below, the functions $\{\lambda_j\}$ that appear in this factorization correspond to a solution of Kawamuko's system.) Then the following proposition holds:

Proposition 2.3. *For q_1, q_2 given by* (2.24) *and* (2.25), *and for $\{\lambda_j\}$ defined by* (2.27), *we have*

$$q_1 = -\left(\frac{2}{\gamma}x^m + \frac{2}{\gamma}C(x) + 2t + \frac{2\theta_2 - \gamma}{\gamma x}\right) - \sum_{j=1}^{m}\frac{1}{x - \lambda_j/2}, \tag{2.28}$$

$$q_2 = \frac{2(\theta_1 + \theta_2)}{\gamma^2}x^{m-1} + \frac{1}{x}L(x) + \sum_{j=1}^{m}\frac{\lambda_j\mu_j}{x(x - \lambda_j/2)}, \tag{2.29}$$

where $L(x)$ is a polynomial in x of degree $m-1$, and $\{\mu_j\}$ are defined by

$$\lambda_j\mu_j = -\frac{1}{\gamma}\big[V(\lambda_j/2) - \theta_1 - \theta_2\big]. \tag{2.30}$$

Remark 2.2. (2.27) and (2.30) describe how a solution $\{u_j, v_j\}$ of $(P_{\mathrm{IV}})_m$ is related to a solution $\{\lambda_j, \mu_j\}$ of (1.42).

Let us prove Proposition 2.3. It is easy to see that (2.28) follows from (2.24). We next prove (2.29). First, we note

$$\frac{V(x) - \theta_1 - \theta_2}{\gamma x\widetilde{A}_{1,2}}\frac{\partial\widetilde{A}_{1,2}}{\partial x} = \sum_{j=1}^{m}\frac{V(x) - V(\lambda_j/2)}{\gamma x(x - \lambda_j/2)} + \sum_{j=1}^{m}\frac{V(\lambda_j/2) - \theta_1 - \theta_2}{\gamma x(x - \lambda_j/2)}. \tag{2.31}$$

Hence (2.30) entails

$$q_2 = \frac{1}{\gamma^2 x^2}\det\widetilde{A} + \frac{1}{\gamma x}\frac{\partial V}{\partial x} - \sum_{j=1}^{m}\frac{V(x) - V(\lambda_j/2)}{\gamma x(x - \lambda_j/2)} + \sum_{j=1}^{m}\frac{\lambda_j\mu_j}{x(x - \lambda_j/2)}. \tag{2.32}$$

To show that q_2 is of the required form we prepare the following:

Lemma 2.1. *(i) For $U(x)$, $V(x)$, $W(x)$, $C(x)$ given in* (1.34) *and* (2.4)*, where $\{w_j\}$ are given in* (1.3)*, we have*

$$x^{m+1}V(x) - U(x)\big[xV(x) + W(x)\big] + \frac{1}{2}V(x)^2 - C(x)W(x) = R(x), \tag{2.33}$$

where

$$R(x) = \sum_{n=1}^{m} x^{m-n}\left(\sum_{j=n}^{m} u_{n+m-j}w_j + \frac{1}{2}\sum_{j=n}^{m} v_{n+m-j}v_j - \sum_{j=n}^{m} c_{n+m-j}w_j\right)$$
$$+ \sum_{n=1}^{m-1} x^{m-n} \sum_{j=n+1}^{m} u_{n+m-j+1}v_j. \tag{2.34}$$

(ii) For the matrix $\widetilde{A}$ given in (2.21)*, the following hold:*

(a) $\det\widetilde{A}$ *is a polynomial in x of degree $m+1$ whose highest degree term is $2(\theta_1 + \theta_2)x^{m+1}$.*

(b) $\det\widetilde{A}\big|_{x=0} = 0$.

Remark 2.3. Note that in Lemma 2.1 (i) we do not assume that $\{u_j, v_j\}$ is a solution of $(P_{\mathrm{IV}})_m$. This Lemma 2.1 (i) will be used in the proof of Theorem 1.3, i.e., the proof of our main theorem for $(P_{\mathrm{II}})_m$.

Proof. (i) Using the definition of $U(x)$, $V(x)$, $W(x)$ and $C(x)$, we find

$$\sum_{j=1}^{m} u_j x^{m-j}\cdot W(x) + x\sum_{j=1}^{m} u_j x^{m-j}\cdot V(x) + \frac{1}{2}V(x)^2 - C(x)W(x)$$
$$= x^m u_1 v_1 + x^m \sum_{n=2}^{m}\left\{\sum_{j=1}^{m-1}(u_{n-j}w_j + \frac{1}{2}v_{n-j}v_j - c_{n-j}w_j) + \sum_{j=1}^{n} u_{n-j+1}v_j\right\}x^{m-n}$$
$$+ R(x). \tag{2.35}$$

The left-hand side of (2.35) is

$$(x^m - U(x))W(x) + x(x^m - U(x))V(x) + \frac{1}{2}V(x)^2 - C(x)W(x). \tag{2.36}$$

By using (1.3) we find that the right-hand side of (2.35) becomes

$$x^m u_1 v_1 + x^m(W(x) - w_1 x^{m-1}) + R(x), \tag{2.37}$$

Thus (i) follows.

(ii) By a straightforward computation, we have

$$\det\widetilde{A} = -(V - \theta_1 - \theta_2)(2x^{m+1} + V + 2xC(x) + 2\gamma xt - \theta_1 + \theta_2)$$

$$
\begin{aligned}
&+2(U+C(x)+\gamma t)(xV+W+v_{m+1}+\gamma t v_1)\\
=\ &(\theta_1+\theta_2)(2x^{m+1}+2xC(x)+2\gamma xt-\theta_1+\theta_2)\\
&+2\theta_1 V+\gamma t(-xV+W+v_1U+v_1C(x)+v_{m+1}+\gamma t v_1)\\
&-\Big[2x^{m+1}V+V^2-2U(xV+W)-2C(x)W\Big]. \qquad (2.38)
\end{aligned}
$$

Then, applying (i), we obtain

$$
\begin{aligned}
\det\widetilde{A}=\ &(\theta_1+\theta_2)(2x^{m+1}+2xC(x)+2\gamma xt-\theta_1+\theta_2)+2R(x)\\
&+2\theta_1 V+\gamma t(-xV+W+v_1U+v_1C(x)+v_{m+1}+\gamma t v_1). \qquad (2.39)
\end{aligned}
$$

This proves (a). Finally, we note that (1.10) entails

$$
\begin{aligned}
\det\widetilde{A}\Big|_{x=0}&=-(v_m-\theta_1-\theta_2)(v_m-\theta_1+\theta_2)\\
&\quad+2(-u_m+c_m+\gamma t)(w_m+v_{m+1}+\gamma t v_1)\\
&=-(v_m-\theta_1-\theta_2)(v_m-\theta_1+\theta_2)+(v_m-\theta_1)^2-{\theta_2}^2\\
&=0. \qquad (2.40)
\end{aligned}
$$

Thus the proof of Lemma 2.1 is completed. □

Now let us return to the proof of Proposition 2.3. It follows from Lemma 2.1 and (2.32) that

$$
q_2(x)=\frac{2(\theta_1+\theta_2)}{\gamma^2}x^{m-1}+\frac{1}{x}L(x)+\sum_{j=1}^{m}\frac{\lambda_j\mu_j}{x(x-\lambda_j/2)}, \qquad (2.41)
$$

where

$$
L(x)=\frac{1}{\gamma^2 x}\left(\det\widetilde{A}-2(\theta_1+\theta_2)x^{m+1}\right)+\frac{1}{\gamma}\frac{\partial V}{\partial x}-\sum_{j=1}^{m}\frac{V(x)-V(\lambda_j/2)}{\gamma(x-\lambda_j/2)} \qquad (2.42)
$$

is a polynomial in x of degree $m-1$ as is required in Proposition 2.3. Thus we have completed the proof of Proposition 2.3.

2.3 The relation between Kawamuko's system and the compatibility conditions of (2.23a) and (2.23b).

In this subsection we show the following:

Proposition 2.4. *We assume that $\{\lambda_j\}$ given by* (2.27) *are mutually distinct. Then the compatibility conditions of the Lax pair* (2.23a) *and* (2.23b) *with q_1, q_2 being expressed as in* (2.28) *and* (2.29), *and with $\mathcal{C}$ and $\mathcal{D}$ being given by* (2.26), *are expressed by the Hamiltonian system* (1.42) *for K given by* (1.40).

Proof. We first note that $x = \lambda_j/2$ for $j = 1, 2, \cdots, m$ are apparent singular points because (2.23a) comes from (2.20): $\vec{\varphi}$ is holomorphic near $x = \lambda_j/2$, and hence φ is not logarithmic there. We change the independent variable x to $z = 2x$ in (2.23a) and (2.23b). We then obtain

$$\left(\frac{\partial^2}{\partial z^2} + \widetilde{q}_1(z)\frac{\partial}{\partial z} + \widetilde{q}_2(z)\right)\varphi = 0, \tag{2.43a}$$

$$\frac{\partial \varphi}{\partial t} = 2\mathcal{C}\frac{\partial \varphi}{\partial z} + \mathcal{D}\varphi \tag{2.43b}$$

with

$$\widetilde{q}_1(z) = \frac{1}{2}q_1(\frac{1}{2}z) = -\left[\frac{1}{2^m\gamma}z^m + \frac{1}{\gamma}C(\frac{1}{2}z) + t + \frac{2\theta_2 - \gamma}{\gamma z}\right] - \sum_{j=1}^{m}\frac{1}{z-\lambda_j}, \tag{2.44}$$

$$\widetilde{q}_2(z) = \frac{1}{4}q_2(\frac{1}{2}z) = \frac{\theta_1+\theta_2}{2^m\gamma^2}z^{m-1} + \frac{1}{2z}L(\frac{1}{2}z) + \sum_{j=1}^{m}\frac{\lambda_j\mu_j}{z(z-\lambda_j)}, \tag{2.45}$$

$$\mathcal{C} = \frac{2^{m-1}\gamma z}{\prod_{j=1}^{m}(z-\lambda_j)}. \tag{2.46}$$

Let us define

$$\overline{p}_j = p_j(z,t)\Bigg|_{\begin{cases} t_1 = t + 2^m c_m, \\ t_k = 2^{m-k+1}c_{m-k+1} \quad (2 \le k \le m)\end{cases}} \tag{2.47}$$

for $j = 1, 2$, where p_j $(j = 1, 2)$ are given by (1.26) and (1.27) with $g = m$. Then we can readily find that, if we set γ, θ_j $(j = 1, 2)$ as in (1.43), $\widetilde{q}_1$ and $\widetilde{q}_2$ respectively coincide with $\overline{p}_1$ and $\overline{p}_2$ except for the term $L(z/2)/(2z)$ in $\widetilde{q}_2$, $-\sum_{k=1}^{g} h_{g+1-k}x^{k-2}$ in $\overline{p}_2$. But recall that these terms are uniquely determined by the requirement that each of $z = \lambda_j$ for $j = 1, 2, \cdots, m$ should be a non-logarithmic singular point. Hence we conclude

$$L(\frac{1}{2}z) = -2\sum_{j=1}^{m} h_j z^{m-j} \tag{2.48}$$

with h_j given by (1.28). Therefore we obtain $\widetilde{q}_j = \overline{p}_j$ for $j = 1, 2$. Furthermore we can verify that $2\mathcal{C} = \mathcal{A}_1$ with $\mathcal{A}_1$ given by (1.33). Since the Hamiltonian system (1.42) is a way to write down (A.16) with $x = z$, $p = \overline{p}_1$, $q = \overline{p}_2$, $\mathcal{A} = \mathcal{A}_1$, the fact that $\overline{p}_j = \widetilde{q}_j$ $(j = 1, 2)$ and $\mathcal{A}_1 = 2\mathcal{C}$ guarantees that the compatibility conditions of (2.43a) and (2.43b) give the Hamiltonian system (1.42). Thus the proof is completed. □

Proof of Theorem 1.4. Proposition 2.4 asserts that the compatibility conditions of the Lax pair (2.23a) and (2.23b) expressed in terms of $\{\lambda_j, \mu_j\}$ given in (1.41) are nothing but Kawamuko's system restricted to a complex line $\{\vec{t} = (t_1, \cdots, t_m); t_1 = t + 2^m c_m,\ t_k = 2^{m-k+1}c_{m-k+1}\ (2 \le k \le m)\}$ if the constants κ_0 and κ_∞ are chose as in (1.43). On the other hand, Proposition 2.4 asserts that the compatibility conditions of

(2.23a) and (2.23b) are the same as those of (2.20). The latter one is clearly same as the compatibility conditions of (2.1), while the compatibility conditions of (2.1) expressed in terms of $\{u_j, v_j\}$ are nothing by $(P_{\text{IV}})_m$. This completes the proof of Theorem 1.4. □

3 The proof of the main theorem for P_{II}-hierarchy

As the proof of Theorem 1.3 is more or less the same as that of Theorem 1.4, we content ourselves with giving its essential points here. First the Lax pair for $(P_{\text{II}})_m$ is given as follows:

Theorem 3.1. *The compatibility conditions of the following equations* (3.1) *are equivalent to* (1.1)*:*

$$\gamma \frac{\partial \vec{\psi}}{\partial x} = A\vec{\psi}, \quad \frac{\partial \vec{\psi}}{\partial t} = B\vec{\psi}, \tag{3.1}$$

where

$$A = \begin{pmatrix} -\left[x^{m+1} + V + xC(x) + \gamma t\right] & U + C(x) \\ -2\left[xV + W + \kappa\right] & x^{m+1} + V + xC(x) + \gamma t \end{pmatrix}, \tag{3.2}$$

$$B = \begin{pmatrix} -(x + u_1) & 1 \\ -2v_1 & x + u_1 \end{pmatrix} \tag{3.3}$$

with U, V and W being as in (1.34) *and* $C(x) = \displaystyle\sum_{j=1}^{m} c_j x^{m-j}$.

Let us first define Δ_j $(j = 1, 2, 3)$ by

$$\frac{\partial A}{\partial t} - \gamma \frac{\partial B}{\partial x} + AB - BA = \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & -\Delta_1 \end{pmatrix} \tag{3.4}$$

for A and B given by (3.2) and (3.3). Then in parallel with Proposition 2.1, we can prove the following:

Proposition 3.1. *(i) $\Delta_1 = \Delta_2 = 0$ implies $(P_{II})_m$ defined by* (1.1).

(ii) If $\{u_j, v_j\}$ satisfies $(P_{II})_m$, then

$$\frac{dw_j}{dt} - v_1 \frac{du_j}{dt} - u_1 \frac{dv_j}{dt} = 0 \quad \textit{for} \quad 1 \le j \le m.$$

(iii) $\Delta_1 = \Delta_2 = 0$ implies $\Delta_3 = 0$.

Proof. By a straightforward computation we obtain

$$\Delta_1 = \sum_{j=1}^{m} \left[-\frac{dv_j}{dt} + 2\{v_1 u_j + v_{j+1} + w_j\} - 2c_j v_1 \right] x^{m-j}, \tag{3.5}$$

$$\Delta_2 = \sum_{j=1}^{m} \left[-\frac{du_j}{dt} - 2\{u_1 u_j + v_j + u_{j+1}\} + 2c_j u_1 \right] x^{m-j}, \tag{3.6}$$

$$\Delta_3 = 2(x+u_1)\Delta_1 + 2v_1\Delta_2 - 2\sum_{j=1}^{m} \left\{ \frac{dw_j}{dt} - u_1\frac{dv_j}{dt} - v_1\frac{du_j}{dt} \right\} x^{m-j}. \tag{3.7}$$

Hence (i) follows from (3.5) and (3.6). We can prove (ii) using (2.12) together with the induction on j. Then (iii) immediately follows from (ii) and (3.7). □

We next derive an appropriate system of scalar equations from (3.1). We first change the unknown function $\vec{\psi}$ to

$$\vec{\varphi} = \exp\left[\int^{x} \frac{x^{m+1} + xC(x) + \gamma t}{\gamma} dx \right] \vec{\psi}. \tag{3.8}$$

Then $\vec{\varphi}$ satisfies

$$\gamma\frac{\partial\vec{\varphi}}{\partial x} = \widetilde{A}\vec{\varphi}, \quad \frac{\partial\vec{\varphi}}{\partial t} = \widetilde{B}\vec{\varphi} \tag{3.9}$$

with

$$\begin{aligned} \widetilde{A} &= A + (x^{m+1} + C(x) + \gamma t)I_2 \\ &= \begin{pmatrix} -V & U + C(x) \\ -2(xV + W + \kappa) & 2x^{m+1} + V + 2xC(x) + 2\gamma t \end{pmatrix}, \end{aligned} \tag{3.10}$$

$$\widetilde{B} = B + xI_2 = \begin{pmatrix} -u_1 & 1 \\ -2v_1 & 2x + u_1 \end{pmatrix}. \tag{3.11}$$

In parallel with the reasoning in §2.2 we consider the equations that the first component of $\vec{\varphi}$ satisfies, that is,

$$\left(\frac{\partial^2}{\partial x^2} + q_1(x)\frac{\partial}{\partial x} + q_2(x) \right)\varphi = 0, \tag{3.12a}$$

$$\frac{\partial\varphi}{\partial t} = \mathcal{C}\frac{\partial\varphi}{\partial x} + \mathcal{D}\varphi, \tag{3.12b}$$

where

$$q_1 = -\frac{1}{\gamma}\mathrm{tr}\,\widetilde{A} - \frac{1}{\widetilde{A}_{1,2}}\frac{\partial\widetilde{A}_{1,2}}{\partial x}, \quad q_2 = \frac{1}{\gamma^2}\det\widetilde{A} + \frac{1}{\gamma}\frac{\partial V}{\partial x} - \frac{V}{\gamma\widetilde{A}_{1,2}}\frac{\partial\widetilde{A}_{1,2}}{\partial x}, \tag{3.13}$$

$$\mathcal{C} = \frac{\gamma}{\widetilde{A}_{1,2}}, \qquad \mathcal{D} = \frac{V}{\widetilde{A}_{1,2}} - u_1. \tag{3.14}$$

Here we set $\widetilde{A}_{1,2} = U + C(x)$, which is the $(1,2)$-component of the matrix $\widetilde{A}$. By the same argument as in §2.2 we obtain

Proposition 3.2. *We assume that* $\widetilde{A}_{1,2} = U + C(x) = 0$ *has no double roots with respect to* x*. Then* (3.12a) *and* (3.12b) *are compatible if and only if* $\{u_j, v_j\}$ *is a solution of* $(P_{II})_m$*.*

We then factorize $\widetilde{A}_{1,2}$ as

$$\widetilde{A}_{1,2} = U + C(x) = \prod_{j=1}^{m}(x - \frac{1}{2}\lambda_j). \tag{3.15}$$

to define $\{\lambda_j\}$. In parallel with Proposition 2.3, we can prove the following by using Lemma 2.1 (i):

Proposition 3.3. *For* q_1*,* q_2 *defined by* (3.13)*, and for* $\{\lambda_j\}$ *defined by* (3.15)*, we have*

$$q_1 = -\frac{2}{\gamma}\left[x^{m+1} + xC(x) + \gamma t\right] - \sum_{j=1}^{m} \frac{1}{x - \lambda_j/2}, \tag{3.16}$$

$$q_2 = \frac{2\kappa}{\gamma^2}x^m + L(x) + \sum_{j=1}^{m} \frac{2\mu_j}{x - \lambda_j/2}, \tag{3.17}$$

where $L(x)$ *is a polynomial of degree* $m - 1$*, and* $\{\mu_j\}$ *are defined by*

$$\mu_j = -\frac{1}{2\gamma}V(\lambda_j/2). \tag{3.18}$$

To compare (3.12a) and (3.12b) with the linear equation (1.11) associated with the A_g-system, we change the independent variables x to $z = 2x$. We then obtain (2.43a) and (2.43b) with

$$\widetilde{q}_1(z) = \frac{1}{2}q_1(\frac{1}{2}z) = -\left[\frac{1}{2^{m+1}\gamma}z^{m+1} + \frac{1}{2\gamma}zC(\frac{1}{2}z) + t\right] - \sum_{j=1}^{m} \frac{1}{z - \lambda_j}, \tag{3.19}$$

$$\widetilde{q}_2(z) = \frac{1}{4}q_2(\frac{1}{2}z) = \frac{\kappa}{2^{m+1}\gamma^2}z^m + \frac{1}{4}L(\frac{1}{2}z) + \sum_{j=1}^{m} \frac{\mu_j}{z - \lambda_j}, \tag{3.20}$$

$$\mathcal{C} = \frac{2^m\gamma}{\prod_{j=1}^{m}(z - \lambda_j)}. \tag{3.21}$$

Then, choosing γ and κ as in (1.38), we can show that

$$\widetilde{q}_j = p_j(z,t)\Bigg|_{\begin{cases} t_1 = t, \\ t_k = 2^{m-k+3}c_{m-k+2}/k \quad (2 \le k \le m) \end{cases}}, \quad \mathcal{A}_1 = 2\mathcal{C}, \tag{3.22}$$

where p_1, p_2 and $\mathcal{A}_1$ are respectively defined by (1.12), (1.13) and (1.24). Since the Hamiltonian system (1.37), where K is defined by (1.35), is given as the compatibility condition (A.16) with $x = z$, $p = \overline{p}_1$, $q = \overline{p}_2$, $\mathcal{A} = \mathcal{A}_1$, we obtain the following proposition:

Proposition 3.4. *We assume that* $\{\lambda_j\}$ *given by* (3.15) *are mutually distinct. Then the compatibility conditions of the Lax pair* (3.12a) *and* (3.12b) *with* q_1, q_2 *being expressed as in* (3.16) *and* (3.17)*, and with* $\mathcal{C}$ *and* $\mathcal{D}$ *being given in* (3.14)*, are expressed by the Hamiltonian system* (1.37) *for* K *given by* (1.35)*.*

Theorem 1.3 now follows from Proposition 3.2 and Proposition 3.4.

Appendix

A Relations between the compatibility conditions of matrix equations and those of the associated scalar equations.

Let us first consider a system of differential equations

$$\frac{\partial\vec{\psi}}{\partial x} = A\vec{\psi}, \quad \frac{\partial\vec{\psi}}{\partial t} = B\vec{\psi} \tag{A.1}$$

for $\vec{\psi} = {}^t(\psi_1, \psi_2)$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{A.2}$$

Then the compatibility conditions of (A.1) are given by $\Theta_j = 0$ $(j = 1, 2, 3, 4)$, where

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + AB - BA = \begin{pmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{pmatrix}. \tag{A.3}$$

Explicit form of Θ_j can be readily found:

$$\Theta_1 = \frac{\partial a}{\partial t} - \frac{\partial \alpha}{\partial x} + b\gamma - c\beta, \qquad \Theta_2 = \frac{\partial b}{\partial t} - \frac{\partial \beta}{\partial x} + (a-d)\beta - b(\alpha - \delta), \tag{A.4}$$

$$\Theta_3 = \frac{\partial c}{\partial t} - \frac{\partial \gamma}{\partial x} - (a-d)\gamma + c(\alpha - \delta), \quad \Theta_4 = \frac{\partial d}{\partial t} - \frac{\partial \delta}{\partial x} + c\beta - b\gamma. \tag{A.5}$$

We first consider how the compatibility condition changes through the transformation of the unknown function by $\vec{\psi} = G\varphi$ with a regular matrix G. It is easy to confirm that $\vec{\varphi}$ satisfies the following:

$$\frac{\partial\vec{\varphi}}{\partial x} = \widetilde{A}\vec{\varphi}, \quad \frac{\partial\vec{\varphi}}{\partial t} = \widetilde{B}\vec{\varphi} \tag{A.6}$$

with

$$\widetilde{A} = G^{-1}AG - G^{-1}\frac{\partial G}{\partial x}, \quad \widetilde{B} = G^{-1}BG - G^{-1}\frac{\partial G}{\partial t}. \tag{A.7}$$

Then a straightforward computation gives

Proposition A.1. *For $\widetilde{A}$, $\widetilde{B}$ defined by* (A.7), *we obtain*

$$\frac{\partial \widetilde{A}}{\partial t} - \frac{\partial \widetilde{B}}{\partial x} + \widetilde{A}\widetilde{B} - \widetilde{B}\widetilde{A} = G^{-1}\Big[\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + AB - BA\Big]G = G^{-1}\begin{pmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{pmatrix} G. \quad \text{(A.8)}$$

We next consider the relation between the compatibility conditions of (A.1) and those of the associated scalar equations (A.9a) and (A.9b) below. We first recall the following:

Proposition A.2. *Assume b does not vanish identically. Then the first component ψ_1 of a solution $\vec{\psi}$ of* (A.1) *satisfies the following differential equations:*

$$\left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right)\psi = 0, \tag{A.9a}$$

$$\frac{\partial \psi}{\partial t} = \mathcal{A}\frac{\partial \psi}{\partial x} + \mathcal{B}\psi \tag{A.9b}$$

with

$$p = -\mathrm{tr}A - \frac{1}{b}\frac{\partial b}{\partial x}, \qquad q = \det A - \frac{\partial a}{\partial x} + \frac{a}{b}\frac{\partial b}{\partial x}, \tag{A.10}$$

$$\mathcal{A} = \frac{\beta}{b}, \qquad \mathcal{B} = \alpha - \frac{a}{b}\beta. \tag{A.11}$$

We note that the compatibility conditions of scalar equations (A.9a) and (A.9b) are well-known: $\overline{\Theta}_1 = \overline{\Theta}_2 = 0$, where

$$\overline{\Theta}_1 = \frac{\partial q}{\partial t} + \frac{\partial^2 \mathcal{B}}{\partial x^2} + p\frac{\partial \mathcal{B}}{\partial x} - 2q\frac{\partial \mathcal{A}}{\partial x} - \frac{\partial q}{\partial x}\mathcal{A}, \tag{A.12}$$

$$\overline{\Theta}_2 = \frac{\partial p}{\partial t} + 2\frac{\partial \mathcal{B}}{\partial x} + \frac{\partial^2 \mathcal{A}}{\partial x^2} - p\frac{\partial \mathcal{A}}{\partial x} - \frac{\partial p}{\partial x}\mathcal{A}. \tag{A.13}$$

(See, e.g., [O, §1.3].)

Remark A.1. (A.12) can be written as follows:

$$\begin{aligned}\overline{\Theta}_1 = {} & \frac{1}{2}\frac{\partial \overline{\Theta}_2}{\partial x} + \frac{1}{2}p\overline{\Theta}_2 \\ & - \frac{1}{2}\left\{\frac{\partial^3 \mathcal{A}}{\partial x^3} + \left(4q - p^2 - 2\frac{\partial p}{\partial x}\right)\frac{\partial \mathcal{A}}{\partial x} + \left(-\frac{\partial^2 p}{\partial x^2} - p\frac{\partial p}{\partial x} + 2\frac{\partial q}{\partial x}\right)\mathcal{A}\right. \\ & \qquad \left. + \frac{\partial^2 p}{\partial x \partial t} + p\frac{\partial p}{\partial t} - 2\frac{\partial q}{\partial t}\right\}.\end{aligned} \tag{A.14}$$

Hence, as is shown in [O, §1.3], $\overline{\Theta}_1 = \overline{\Theta}_2 = 0$ is equivalent to

$$\frac{\partial \mathcal{B}}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left(p\mathcal{A} - \frac{\partial \mathcal{A}}{\partial x}\right) - \frac{1}{2}\frac{\partial p}{\partial t}, \tag{A.15}$$

$$\frac{\partial^3 \mathcal{A}}{\partial x^3} - 4Q\frac{\partial \mathcal{A}}{\partial x} - 2\frac{\partial Q}{\partial x}\mathcal{A} + 2\frac{\partial Q}{\partial t} = 0 \quad \text{with} \quad Q = -q + \frac{1}{4}p^2 + \frac{1}{2}\frac{dp}{dx}. \tag{A.16}$$

Proposition A.3. *Let* $\{\Theta_j\}$ *be as in* (A.3) *and let* $\overline{\Theta}_1$ *and* $\overline{\Theta}_2$ *be given by* (A.12) *and* (A.13)*, where* p*,* q*,* $\mathcal{A}$ *and* $\mathcal{B}$ *are given by* (A.10) *and* (A.11)*. We further assume that* b *does not vanish identically. Then we have*

$$\overline{\Theta}_1 + a\overline{\Theta}_2 = -\left[\frac{d\Theta_1}{dx} + (a - d - \frac{1}{b}\frac{\partial b}{\partial x})\Theta_1 + c\Theta_2 + b\Theta_3\right], \tag{A.17}$$

$$\overline{\Theta}_2 = -\left[\frac{d}{dx}\left(\frac{\Theta_2}{b}\right) + \Theta_1 + \Theta_4\right]. \tag{A.18}$$

Proof. We first change the unknown function $\vec{\psi}$ of (A.1) by $\vec{\psi} = G\vec{\varphi}$, where

$$G = \frac{1}{b}\begin{pmatrix} b & 0 \\ -a & 1 \end{pmatrix}. \tag{A.19}$$

Then $\vec{\varphi}$ satisfies (A.6) with

$$\widetilde{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix}, \tag{A.20}$$

where

$$\widetilde{\alpha} = \alpha - \frac{a}{b}\beta, \quad \widetilde{\beta} = \frac{\beta}{b}, \tag{A.21}$$

and $\widetilde{\gamma}$ and $\widetilde{\delta}$ are some functions of the components of A and B. The compatibility conditions of (A.6) with (A.20) are, then, given by $\widetilde{\Theta}_j = 0$ ($j = 1, 2, 3, 4$), where

$$\frac{\partial \widetilde{A}}{\partial t} - \frac{\partial \widetilde{B}}{\partial x} + \widetilde{A}\widetilde{B} - \widetilde{B}\widetilde{A} = \begin{pmatrix} \widetilde{\Theta}_1 & \widetilde{\Theta}_2 \\ \widetilde{\Theta}_3 & \widetilde{\Theta}_4 \end{pmatrix}. \tag{A.22}$$

By a straightforward computation we find:

$$\widetilde{\Theta}_1 = -\frac{\partial \widetilde{\alpha}}{\partial x} + \widetilde{\gamma} + q\widetilde{\beta}, \tag{A.23}$$

$$\widetilde{\Theta}_2 = -\frac{\partial \widetilde{\beta}}{\partial x} - \widetilde{\alpha} + p\widetilde{\beta} + \widetilde{\delta}, \tag{A.24}$$

$$\widetilde{\Theta}_3 = -\frac{\partial q}{\partial t} - \frac{\partial \widetilde{\gamma}}{\partial x} - q(\widetilde{\alpha} - \widetilde{\delta}) - p\widetilde{\gamma}, \tag{A.25}$$

$$\widetilde{\Theta}_4 = -\frac{\partial p}{\partial t} - \frac{\partial \widetilde{\delta}}{\partial x} - q\widetilde{\beta} - \widetilde{\gamma}. \tag{A.26}$$

From (A.23) and (A.24), we obtain

$$\widetilde{\gamma} = \widetilde{\Theta}_1 + \frac{\partial \widetilde{\alpha}}{\partial x} - q\widetilde{\beta}, \quad \widetilde{\delta} = \widetilde{\Theta}_2 + \widetilde{\alpha} - p\widetilde{\beta} + \frac{\partial \widetilde{\beta}}{\partial x}. \tag{A.27}$$

Substituting (A.27) into (A.25) and (A.26), then using (A.21), we obtain

$$\widetilde{\Theta}_3 = \Big[-p\widetilde{\Theta}_1 + q\widetilde{\Theta}_2 - \frac{d\widetilde{\Theta}_1}{dx}\Big] - \Big[\frac{\partial q}{\partial t} + \frac{\partial^2 \widetilde{\alpha}}{\partial x^2} + p\frac{\partial \widetilde{\alpha}}{\partial x} - 2q\frac{\partial \widetilde{\beta}}{\partial x} - \frac{\partial q}{\partial x}\widetilde{\beta}\Big]$$

$$= \Big[- p\widetilde{\Theta}_1 + q\widetilde{\Theta}_2 - \frac{d\widetilde{\Theta}_1}{dx} \Big] - \overline{\Theta}_1, \tag{A.28}$$

$$\widetilde{\Theta}_4 = -\Big[\widetilde{\Theta}_1 + \frac{d\widetilde{\Theta}_2}{dx}\Big] - \Big[\frac{\partial p}{\partial t} + 2\frac{\partial \widetilde{\alpha}}{\partial x} + \frac{\partial^2 \widetilde{\beta}}{\partial x^2} - \frac{\partial}{\partial x}(p\widetilde{\beta})\Big]$$

$$= -\Big[\widetilde{\Theta}_1 + \frac{d\widetilde{\Theta}_2}{dx}\Big] - \overline{\Theta}_2. \tag{A.29}$$

On the other hand, from Proposition A.1, we obtain

$$\begin{pmatrix} \widetilde{\Theta}_1 & \widetilde{\Theta}_2 \\ \widetilde{\Theta}_3 & \widetilde{\Theta}_4 \end{pmatrix} = G^{-1} \begin{pmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{pmatrix} G = \begin{pmatrix} \Theta_1 - \frac{b}{a}\Theta_2 & \frac{1}{b}\Theta_2 \\ a\left[\Theta_1 - \frac{a}{b}\Theta_2\right] + b\left[\Theta_3 - \frac{a}{b}\Theta_4\right] & \frac{a}{b}\Theta_2 + \Theta_4 \end{pmatrix}. \tag{A.30}$$

By substituting (A.30) into (A.28) and (A.29), we obtain (A.17) and (A.18). □

Using this proposition, we prove Proposition 2.2 in the following manner:

Proof of Proposition 2.2. We apply Proposition A.3 with $A = (\gamma x)^{-1}\widetilde{A}$ and $B = \widetilde{B}$, where $\widetilde{A}$ and $\widetilde{B}$ are respectively given by (2.21) and (2.22). Let $\{\Theta_j\}$ be given by (A.3), and let $\{\Delta_j\}$ be given by (2.5). We first show that

$$\Theta_j = (\gamma x)^{-1}\Delta_j \quad (1 \le j \le 3), \quad \Theta_4 = -(\gamma x)^{-1}\Delta_1. \tag{A.31}$$

By its definition, we obtain

$$\gamma x \begin{pmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{pmatrix} = \frac{\partial \widetilde{A}}{\partial t} - \gamma x \frac{\partial \widetilde{B}}{\partial x} + \widetilde{A}\widetilde{B} - \widetilde{B}\widetilde{A} \tag{A.32}$$

Then it follows from Proposition A.1 that

$$\frac{\partial \widetilde{A}}{\partial t} - \gamma x \frac{\partial \widetilde{B}}{\partial x} + \widetilde{A}\widetilde{B} - \widetilde{B}\widetilde{A} = G^{-1} \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & -\Delta_1 \end{pmatrix} G = \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & -\Delta_1 \end{pmatrix} \tag{A.33}$$

holds, where G is the following 2×2-matrix defined by

$$G = \exp\left[-\int^x \frac{x^{m+1} + xC(x) + \gamma x t + \theta_2}{\gamma x} dx\right] I_2. \tag{A.34}$$

Here I_2 stands for the 2×2 identity matrix. Note that that (2.20) and (2.1) are related by (A.34) (cf. (2.19)). Thus we obtain (A.31). Then Proposition A.3 entails

$$\overline{\Theta}_1 + a\overline{\Theta}_2 = -\left[\frac{d}{dx}\left(\frac{\Delta_1}{\gamma x}\right)\right] - \frac{1}{\gamma x}\left[(a - d - \frac{1}{b}\frac{\partial b}{\partial x})\Delta_1 + c\Delta_2 + b\Delta_3\right], \tag{A.35}$$

$$\overline{\Theta}_2 = -\frac{d}{dx}\left(\frac{\Delta_2}{\gamma x b}\right), \tag{A.36}$$

where a, b, c and d are defined by

$$\frac{1}{\gamma x}\widetilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{A.37}$$

Now suppose that (2.20) is a compatible system. Then we find $\Delta_j = 0$ for $1 \leq j \leq 3$. Hence it immediately follows from (A.35) and (A.36) that $\overline{\Theta}_1 = \overline{\Theta}_2 = 0$. This means that (2.23a) and (2.23b) are compatible.

Conversely let us suppose that (2.23a) and (2.23b) are compatible, that is, let us assume that $\widetilde{\Theta}_1 = \widetilde{\Theta}_2 = 0$. Then it follows from (A.36) that

$$\Delta_2 = C(t)\gamma x b \tag{A.38}$$

hods for some $C(t)$ free from x. On the other hand it follows from (2.10) that Δ_2 is a polynomial of x with degree at most $(m-1)$, while

$$\gamma x b = \widetilde{A}_{1,2} = U + C(x) + \gamma t \tag{A.39}$$

is a monic polynomial of x with degree m. Therefore (A.38) implies that $C(t)$ should vanish identically, and hence $\Delta_2 = 0$. In order to confirm $\Delta_1 = 0$, we first factorize $\widetilde{A}_{1,2}$ as in (2.27), and the consider the residues of the right-hand side of (A.35) at $x = \lambda_j/2$ $(j = 1, \cdots, m)$. Since Δ_j $(j = 1, 2, 3)$ are holomorphic there, (A.35) together with $\overline{\Theta}_1 = \overline{\Theta}_2 = 0$ entails

$$\Delta_1\big|_{x=\lambda_j/2} = 0 \tag{A.40}$$

for each j. On the other hand (2.9) tells us that Δ_1 is polynomial of x with degree at most $(m-1)$. Since (A.40) implies Δ_1 should vanish at m points, we conclude that $\Delta_1 = 0$. Thus we find $\Delta_1 = \Delta_2 = 0$, and hence Δ_3 also vanishes by Proposition 2.1 (iii). Therefore (2.20) is a compatible system. This completes the proof of Proposition 2.2. □

B Introducing a large parameter.

As our eventual purpose is to apply the results in this paper to the exact WKB analysis of the higher order Painlevé equations, it is preferable that we introduce a large parameter η to all the equations considered in this paper. Although we have not done so here to make the presentation simpler in its appearance, we can really introduce a large parameter η consistently to all the equations, namely, consistently in the Lax pair and in its compatibility conditions, i.e., the Painlevé equation. Here we list up the final result for our future reference. Some detailed explanations will be given in [Ko].

$(P_{\mathrm{II}})_m$ with a large parameter:

$$\begin{cases} \eta^{-1}\dfrac{du_j}{dt} &= -2\Big[u_1 u_j + v_j + u_{j+1}\Big] + 2c_j u_1, \\ \eta^{-1}\dfrac{dv_j}{dt} &= 2\Big[v_1 u_j + v_{j+1} + w_j\Big] - 2c_j v_1. \end{cases} \quad (1 \leq j \leq m). \tag{B.1}$$

$$\text{with} \quad u_{m+1} = \gamma t, \quad v_{m+1} = \kappa. \tag{B.2}$$

Here $\{w_m\}$ are given by (1.3). Its Lax pair is given by

$$\gamma \frac{\partial \vec{\psi}}{\partial x} = \eta A \vec{\psi}, \quad \frac{\partial \vec{\psi}}{\partial t} = \eta B \vec{\psi}, \tag{B.3}$$

where A and B are given by (3.2) and (3.3).

A_g-systems with a large parameter:

$$\frac{\partial \lambda_j}{\partial t_k} = \eta \frac{\partial H_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\eta \frac{\partial H_k}{\partial \lambda_j}, \tag{B.4}$$

where H_j are given by (1.20) with

$$h_j = \frac{1}{2} \sum_{k=1}^{g} \left[N_k N^{j,k} {\mu_k}^2 - U_{j,k} \mu_k - (2\alpha+1) N_k N^{j,k} {\lambda_k}^g \right], \tag{B.5}$$

$$U_{j,k} = N_k N^{j,k} A_g(\lambda_k, t) + \eta^{-1} \sum_{\substack{l=1,2,\cdots,g \\ l \neq k}} \frac{N_k N^{j,k} + N_l N^{j,l}}{\lambda_k - \lambda_l}. \tag{B.6}$$

Here we note that

$$U_{j,k} = N_k N^{j,k} A_g(\lambda_k, t) - \eta^{-1} N_k \sum_{p=0}^{j-2} (-1)^p e_p^{(k)} \, {\lambda_k}^{j-p-2}. \tag{B.7}$$

The associated linear equations is

$$\frac{d^2 y}{dz^2} + \eta p_1(z,t) \frac{dy}{dz} + \eta^2 p_2(z,t) y = 0, \tag{B.8}$$

where

$$p_1(z,t) = -A_g(z,t) - \eta^{-1} \sum_{k=1}^{g} \frac{1}{z - \lambda_k}, \tag{B.9}$$

$$p_2(z,t) = -(2\alpha+1) z^g - 2 \sum_{j=1}^{g} h_j z^{g-j} + \eta^{-1} \sum_{k=1}^{g} \frac{\mu_k}{z - \lambda_k} \tag{B.10}$$

with $A_g(z,t)$ given in (1.12).

$(P_{\mathrm{IV}})_m$ with a large parameter:

$$\left\{ \begin{array}{rcl} \eta^{-1} \dfrac{du_j}{dt} & = & -2\Big[u_1 u_j + v_j + u_{j+1} \Big] + 2 c_j u_1, \\ \eta^{-1} \dfrac{dv_j}{dt} & = & 2\Big[v_1 u_j + v_{j+1} + w_j \Big] - 2 c_j v_1. \end{array} \right. \quad (1 \leq j \leq m). \tag{B.11}$$

$$\text{with} \quad u_{m+1} = -(\gamma t u_1 + \theta_1 + \frac{1}{2}\eta^{-1}\gamma), \tag{B.12}$$

$$v_{m+1} = -w_m - \gamma t v_1 - \frac{(v_m - \theta_1)^2 - {\theta_2}^2}{2(u_m - \gamma t - c_m)}. \tag{B.13}$$

Its Lax pair is given by

$$\gamma x \frac{\partial \vec{\psi}}{\partial x} = \eta A \vec{\psi}, \quad \frac{\partial \vec{\psi}}{\partial t} = \eta B \vec{\psi}, \tag{B.14}$$

where A and B are given by (2.2) and (2.3).

Kawamuko's system with a large parameter:

$$\frac{\partial \lambda_j}{\partial t_k} = \eta \frac{\partial H_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\eta \frac{\partial H_k}{\partial \lambda_j}, \tag{B.15}$$

where H_j are given by (1.31) with

$$\begin{aligned} h_{j+1} = (-1)^j \sum_{l=1}^{g} \frac{e_j^{(l)}}{\Lambda'(\lambda_l)} \left\{ \lambda_l \mu_l^2 - \left(\sum_{k=1}^{g+1} t_k \lambda_l^k + \kappa_0 \right) \mu_l + \kappa_\infty \lambda_l^g \right\} \\ - \eta^{-1} \sum_{l=1}^{g} \frac{\mu_l}{\Lambda'(\lambda_l)} \sum_{k=0}^{j-1} (-1)^k e_k^{(l)} \lambda_l^{j-k}. \end{aligned} \tag{B.16}$$

The associated linear equations is

$$\frac{d^2 y}{dz^2} + \eta p_1(z,t) \frac{dy}{dz} + \eta^2 p_2(z,t) y = 0, \tag{B.17}$$

where

$$p_1(z,t) = -\sum_{k=0}^{g+1} t_k z^{k-1} - \eta^{-1} \sum_{k=1}^{g} \frac{1}{z - \lambda_k} \quad (t_{g+1} = 1,\ t_0 = \kappa_0 - 1), \tag{B.18}$$

$$p_2(z,t) = \kappa_\infty z^{g-1} - \frac{1}{z} \sum_{k=1}^{g} h_{g+1-k} z^{k-1} + \eta^{-1} \sum_{k=1}^{g} \frac{\lambda_k \mu_k}{z(z - \lambda_k)}. \tag{B.19}$$

References

[AKT] T. Aoki, T. Kawai and Y. Takei: WKB analysis of Painlevé transcendents with a large parameter. II, Structure of Solutions of Differential Equations, World Scientific, 1996, pp. 1–49.

[GJP] P.R. Gordoa, N. Joshi and A. Pickering: On a generalized $2+1$ dispersive water wave hierarchy, Publ. RIMS, Kyoto Univ., **37**(2001), 327–347.

[JMU] M. Jimbo, T. Miwa and K. Ueno: Monodoromy preserving deformation of linear ordinary differential equations with rational coefficients, I, Physica **2D** (1981), 306–352.

[KKNT] T. Kawai, T. Koike, Y. Nishikawa and Y. Takei: On the Stokes geometry of higher order Painlevé equations, Astérisque, **297** (2004), 117–166.

[KT] T. Kawai and Y. Takei: Announcement of the Toulouse Project Part 2. RIMS Koukyuuroku No. 1397, 2004, pp. 172–174.

[Kwm1] H. Kawamuko: Studies on the fourth Painlevé equation in several variables. Thesis, University of Tokyo, 1997.

[Kwm2] ______: On the holonomic deformation of linear differential equations, Proc. Japan Acad. Ser. A Math. Sci., **73** (1997), 152–154.

[Kwm3] ______: On the polynomial Hamiltonian structure associated with the $L(1, g+2; g)$ type, Proc. Japan Acad. Ser. A Math. Sci., **73** (1997), 155–157.

[Kwm4] ______: On the holonomic deformation of linear differential equations with a regular singular point and an irregular singular point, Kyushu J. Math. **57** (2003), no. 1, 1–28.

[Ko] T. Koike: On some representations of Painlevé hierarchies. In preparation.

[L1] D. Liu: Holonomic deformation of linear differential equations of A_g type and polynomial Hamiltonian Structure. Thesis, Univ. of Tokyo, 1997.

[L2] ______: On the holonomic deformation of linear differential equations of A_4 type, Kyushu J. Math., **51** (1997), 393–412.

[L3] ______: Holonomic deformation of linear differential equations of the A_3 type, J. Math. Sci. Univ. Tokyo, **5** (1998), 435–458.

[L4] ______: Polynomial Hamiltonian structure for the A_4-system, Kyushu J. Math., **52** (1998), 455–473.

[L5] ______: On the holonomic deformation of linear differential equations of the A_g type, J. Math. Sci. Univ. Tokyo, **8**(2001), 559–594.

[LO1] D. Liu and K.Okamoto: Holonomic deformation of linear differential equations of the A_3 type and polynomial Hamiltonian structure, Painlevé functions and asymptotic analysis (Japanese) (Kyoto,1995). RIMS Koukyuuroku No. 931, 1995, pp. 20–33.

[LO2] ______: Polynomial Hamiltonian structure for the A_3 system, Kumamoto J. Math. **10** (1997), 31–43.

[MM] M. Mazzocco and M. Y. Mo: The Hamiltonian structure of the second painleve hierarchy, `http://arXiv.org/abs/nlin.SI/0610066`.

[O] K. Okamoto: Isomonodromic deformation and Painlevé equations, and the Garnier systems, J. Fac. Sci. Univ. Tokyo, Sect. IA, **33**(1986), 575–618.

[Tks] K. Takasaki: Hamiltonian structure of PI hierarchy, `http://arXiv.org/abs/nlin.SI/0610073`.

[T] Y.Takei: Instanton-type formal solutions for the first Painlevé hierarchy. To appear in the Proceedings of the Conference "Algebraic Analysis of Differential Equations" in honor of Prof. T. Kawai.